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The asymptotic number of attractors in the random map model

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Abstract

The random map model is a deterministic dynamical system in a finite phase space with n points. The map that establishes the dynamics of the system is constructed by randomly choosing, for every point, another one as its image. We derive here explicit formulae for the statistical distribution of the number of attractors in the system. As in related results, the number of operations involved by our formulae increases exponentially with n; therefore, they are not directly applicable to study the behaviour of systems where n is large. However, our formulae can be used to derive useful asymptotic expressions, as we show.

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1. Introduction

Since the 1970s the random map model has attracted the attention of physicists [1, 2]. Indeed, in 1987 Derrida and Flyvbjerg showed that the random map model is equivalent to the Kauffman model of cellular automata when the number of connections among the automata goes to infinity [3, 4]. This enlarged its application in the realm of theoretical biology, disordered systems and cellular automata, for possible approaches of DNA replication, cell differentiation and evolution theory [1].

On the other hand, half a century ago, some mathematicians approached the random map model in the context of random graphs. First, in 1953, Metropolis and Ulam posed the problem of determining the number $\Theta(n)$ of expected connected components (i.e., attractors) in random graphs with n nodes [5]; at the time, $\Theta(n)$ was estimated to be of order $\log n$. Only one year later, Kruskal elegantly solved the problem obtaining an exact formula together with its corresponding asymptotic behaviour [6]:

$$\Theta(n) = \sum_{k=1}^{n} \frac{n!}{(n-k)!kn^k} \tag{1.1}$$

$$\Theta(n) \approx \frac{1}{2}(\ln 2n + \gamma) + \epsilon \qquad \text{for} \quad n \gg 1 \tag{1.2}$$

where ϵ vanishes for $n \to \infty$, and γ is the Euler–Mascheroni constant.

The statistical distribution of the number of connected components was addressed both by Rubin and Sitgreaves [7], and Folkert [8] in his PhD thesis under the supervision of Leo Katz. Unfortunately, these two practically out of reach works were never published, despite their relevance and an offer by Katz to do so [9]. Later, Harris partially reviewed and enlarged these results, proposing a new combinatorial expression for the distribution of connected components, as well as the complete solution of the simpler case in which the random map is one to one [10]. The mathematical expressions of a general random map found by Folkert and Harris, however correct, are based on constrained sums with a number of terms of order e^n, and involve Stirling numbers of the first kind [11–13]. Therefore, unfortunately, their straightforward use in physics or biology appears quite limited (for example, it is typical to deal with $n \sim 2^{100}$ in models such as those of cellular automata).

In spite of its importance, and as far as we know, a study of the variance for the distribution of connected components has not yet been undertaken.

In this paper we propose a still new combinatorial formula—equivalent to the previous ones, of course—for the statistical distribution of the number of connected components. As in earlier results, it also relies on a constrained sum, and the involved computational effort increases exponentially with n. Nevertheless, it has the advantage of allowing us the derivation of the long-needed asymptotic formula for the statistical distribution. Furthermore, we easily deduce from it asymptotic formulae for the corresponding average and variance.

The paper is organized as follows. Section 2 is devoted to the main definitions of the random map model, and to settle our conventions. Then, in section 3, we determine an exact combinatorial expression for the statistical distribution of the number of connected components in the model. The corresponding asymptotic formula for this distribution is derived in section 4, as well as asymptotic formulae for the average and the variance for the number of connected components. Finally, we present our conclusions in section 5.

2. The random map model and functional graphs

Let $\Omega = \{1, 2, \ldots, n\}$ be a set of n points. To each point in Ω assign at random one point in Ω with uniform probability distribution, thus defining a function $f : \Omega \to \Omega$. In this way a dynamical system has been established on the so-called phase space Ω through the iterations of f; this is the random map model [3, 4].

Since Ω is finite, every orbit of f will eventually end in a periodic attractor, and several questions are in order. For example, what is the expected number of attractors in the system? Which is the statistical distribution of the number of attractors? How large is the dispersion of the distribution? Here, we answer these questions starting from combinatorial arguments.

For each function f on Ω define a *functional graph* whose nodes are precisely the elements of Ω; moreover, if $f(i) = j$ then draw a directed link from node i to node j, and whenever $f(i) = i$ a loop on node i is drawn. As an example, figure 1 shows a functional graph with three connected components (i.e., three attractors) in a set with $n = 11$.

Note that each function f on Ω (or functional graph) can be represented by an $n \times n$ binary matrix $M = \{M_{ij}\}$, where $M_{ij} = 1$ whenever $f(i) = j$ and $M_{ij} = 0$ otherwise. Every row of matrix M has $n - 1$ zeros and one '1'. Clearly, there exist n^n such matrices, and thus n^n is the number of functions f that can be defined on Ω, as well as the number of distinct functional graphs on n nodes.

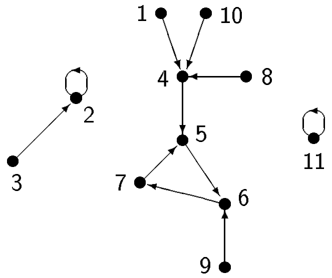

Figure 1. A functional graph with three connected components for $n = 11$.

3. The distribution of connected components

What is the number a_n of connected functional graphs (i.e., having precisely one connected component) that can be found among the n^n functional graphs on n nodes? Through clever combinatorial arguments Katz [9] obtained

$$a_n = (n-1)! \sum_{k=0}^{n-1} \frac{n^k}{k!} \tag{3.1}$$

and showed that, for $n \gg 1$, this figure grows asymptotically as

$$a_n \approx n^n \sqrt{\frac{\pi}{2n}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \tag{3.2}$$

Using equation (3.1) we derive now a new expression for the statistical distribution of the number of connected components in functional graphs. Consider first a partition $\mathcal{P}$ of Ω, in k disjoint subsets $\Omega_1, \ldots, \Omega_k$, with $n_1, \ldots, n_k$ points, respectively. Then

$$\sum_{j=1}^{k} n_j = n \qquad \text{with} \quad 1 \leqslant n_j \leqslant n \quad \text{for} \quad j = 1, \ldots, k. \tag{3.3a}$$

Obviously $\prod_{j=1}^{k} a_{n_j}$ is the number of functional graphs G with k connected components, such that the subset Ω_j defines a connected component of G, for $j = 1, \ldots, k$. Moreover, for given integers $n_1, \ldots, n_k$, satisfying equation (3.3a), the multinomial coefficient $\binom{n}{n_1,\ldots,n_k} = \frac{n!}{n_1!\cdots n_k!}$ yields the number of distinct ways we can distribute n objects in distinguishable boxes $B_1, \ldots, B_k$, of sizes $n_1, \ldots, n_k$, respectively [11, 13]. This leads to

$$\frac{1}{k!} \sum_{\{n_1,\ldots,n_k\}'} \binom{n}{n_1, \ldots, n_k} \prod_{j=1}^{k} a_{n_j}$$

as the number of functional graphs with k connected components, where $\{n_1, \ldots, n_k\}'$ means that the sum is over all vectors $(n_1, \ldots, n_k)$ satisfying equation (3.3a), and the factor $\frac{1}{k!}$ accounts for the unavoidable repetitions incurred by our above assumption of 'distinguishable boxes'.

Therefore, for $k = 1, \ldots, n$, the distribution for the number of connected components can be expressed as

$$\rho_n(k) = \frac{1}{n^n k!} \sum_{\{n_1,\ldots,n_k\}'} \binom{n}{n_1, \ldots, n_k} \prod_{j=1}^{k} a_{n_j}. \tag{3.3b}$$

It is not difficult to see that the sum in equation (3.3*b*) has as many as $\binom{n-1}{k-1}$ terms [13], a figure growing exponentially with n when $k \sim \frac{n}{2}$. This fact dooms to failure any numerical application of equation (3.3*b*) as it stands, and hinders further analytical work.

The approaches of Folkert and Harris lead to increasing complications. The former yields [8, 10]

$$\rho_n(k) = \frac{1}{n^n} \sum_{\mu=k}^{n} \frac{\mathcal{S}_{\mu}^{(k)}}{\mu!} \sum_{\{n_1,\ldots,n_\mu\}''} \binom{n}{n_1,\ldots,n_\mu} n_1^{n_1} \cdots n_\mu^{n_\mu} \tag{3.4a}$$

where $\mathcal{S}_{\mu}^{(k)}$ are the Stirling numbers of the first kind [11–13] (see the appendix), and $\{n_1, \ldots, n_\mu\}''$ means that the sum is over all vectors $n_1, \ldots, n_\mu$, constrained by

$$\sum_{j=1}^{\mu} n_j = n \qquad \text{with} \quad 1 \leqslant n_j \leqslant n. \tag{3.4b}$$

Harris managed to propose [10]

$$\rho_n(k) = \frac{n!}{n^n k!} \sum_{\mu=k}^{n} \mathcal{S}_{\mu}^{(k)} \sum_{\{n_1,\ldots,n_n\}'''} \frac{1}{n_1!,\ldots,n_n!} \left(\frac{1^1}{1!}\right)^{n_1} \left(\frac{2^2}{2!}\right)^{n_2} \cdots \left(\frac{n^n}{n!}\right)^{n_n} \tag{3.5a}$$

in which $\{n_1, \ldots, n_n\}'''$ means that the sum is over all vectors $n_1, \ldots, n_n$, constrained by

$$\sum_{j=1}^{n} n_j = \mu \qquad \text{and} \qquad \sum_{j=1}^{n} j n_j = n \qquad \text{with} \quad 0 \leqslant n_j \leqslant n. \tag{3.5b}$$

Expressions (3.4) and (3.5) appear more difficult to handle than equation (3.3) because of the added extra terms via the summation involving Stirling numbers.

To our knowledge, the statistical moments of the distribution for the number of connected components have not yet been obtained in closed form. Furthermore, the exact calculation performed by Kruskal for the expected number of attractors (1.1) is not derived from a distribution (see [6]). For application purposes ($n \gg 1$) it is then worthwhile to derive manageable asymptotic formulae for $\rho_n(k)$; this is the subject of the next section.

4. Asymptotic expressions

Let us start by defining

$$\beta_m = 2\,\mathrm{e}^{-m} \frac{a_m}{(m-1)!} \tag{4.1}$$

and

$$\alpha_m = 1 - \beta_m. \tag{4.2}$$

Due to the asymptotic relation (3.2) it happens that

$$\beta_m = 1 + \mathcal{O}\left(\frac{1}{m}\right) \qquad \text{for} \quad m \gg 1 \tag{4.3a}$$

and

$$\alpha_m = \mathcal{O}\left(\frac{1}{m}\right) \qquad \text{for} \quad m \gg 1. \tag{4.3b}$$

Now, express equation (3.3*b*) in terms of β_m and use equation (3.3*a*) to obtain

$$\rho_n(k) = \frac{n!\,\mathrm{e}^n}{n^n k! 2^k} \sum_{\{n_1,\ldots,n_k\}'} \prod_{j=1}^{k} \frac{\beta_{n_j}}{n_j}.$$

Constraint (3.3a) can be broken by introducing a Kronecker delta inside the summation. Using the integral representation

$$\delta_{n,m} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{z^m}{z^{n+1}}\, \mathrm{d}z$$

where Γ is any closed contour in the complex plane of z such that the origin is inside, we return to the expression for $\rho_n(k)$ to obtain

$$\rho_n(k) = \frac{n!\,\mathrm{e}^n}{n^n k! 2^k 2\pi \mathrm{i}} \oint_\Gamma \frac{\mathrm{d}z}{z^{n+1}} g_k(z)$$

with a modicum of algebra, where

$$g_k(z) = [\varphi_p(z)]^k \quad \text{with} \quad \varphi_p(z) = \sum_{m=1}^{p} \frac{\beta_m}{m} z^m \quad \text{and} \quad p \geqslant n. \tag{4.4a}$$

Since $g_k(z)$ is an analytical function, we can apply the Cauchy integral theorem to find

$$\rho_n(k) = \frac{\mathrm{e}^n}{n^n k! 2^k} g_k^{(n)}(0) \tag{4.4b}$$

as an alternative way to compute $\rho_n(k)$.

The computation of equation (4.4) may be as difficult as that of equation (3.3); however, the former is manageable for $n \gg 1$, as we shall see. First note that equation (4.4b) does not depend on the particular value we give to p in equation (4.4a), as long as $p \geqslant n$, a feature that we shall use to our convenience. Taking $p = n$ in equation (4.4a), $g_k(z)$ becomes a polynomial of degree nk, and its nth derivative may be calculated in terms of finite differences with the help of Stirling numbers (see formula (A.6) in the appendix), obtaining

$$g_k^{(n)}(0) = n! \sum_{r=n}^{nk} \frac{\mathcal{S}_r^{(n)}}{r!} \Delta^r g_k(0) \tag{4.5a}$$

where $\mathcal{S}_r^{(n)}$ are the Stirling numbers of the first kind, and

$$\Delta^r g_k(0) = \sum_{m=1}^{r} (-1)^{r-m} \binom{r}{m} g_k(m). \tag{4.5b}$$

Now we provide asymptotic approximations for $\varphi_p(z)$ in equation (4.4a) with $p = n$ and $n \gg 1$. Consider first $|z| > 1$, and note that, due to equation (4.3),

$$\varphi_n(z) = \sum_{k=1}^{n} \frac{\beta_k}{k} z^k \approx \sum_{k=1}^{n} \frac{1}{k} z^k \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Now, the Euler–Maclaurin formula states [11, 14]

$$\sum_{k=1}^{n} f(k) \approx \int_1^n f(x)\, \mathrm{d}x + \frac{1}{2}(f(n) + f(1)) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(1)) \tag{4.6a}$$

where $f(x)$ is a C^∞ function over the interval $[1, n]$, and B_k are the Bernoulli numbers defined by the generating function

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \qquad \text{for} \quad |t| < 2\pi \tag{4.6b}$$

with $B_{2k+1} = 0$, for $k = 1, 2, \ldots$.

Let us take $f(x) = \frac{z^x}{x}$ with $|z| > 1$, and substitute in equation (4.6a) obtaining

$$\sum_{k=1}^{n} \frac{\beta_k}{k} z^k \approx \frac{z^n}{n} \left[\frac{1}{\ln z} + \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (\ln z)^{2k-1} \right] \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Using equation (4.6b) yields

$$\sum_{k=1}^{n} \frac{\beta_k}{k} z^k \approx \frac{z}{z-1} \frac{z^n}{n} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \qquad \text{for} \quad 1 < |z| < \mathrm{e}^{2\pi}$$

which can be continued analytically for any $|z| > 1$, thus obtaining

$$\varphi_n(z) = \sum_{m=1}^{n} \frac{\beta_m}{m} z^m \approx \frac{z}{z-1} \frac{z^n}{n} \zeta_n \qquad \text{for} \quad |z| > 1 \tag{4.7a}$$

where

$$\zeta_n = 1 + \mathcal{O}\left(\frac{1}{n}\right). \tag{4.7b}$$

Instead, for $z = 1$, clearly

$$\varphi_n(1) = \sum_{m=1}^{n} \frac{\beta_m}{m} \approx (\ln n + \gamma - \tau_n)\eta_n \tag{4.7c}$$

where

$$\eta_n = 1 + \mathcal{O}\left(\frac{1}{n \ln n}\right) \tag{4.7d}$$

$$\tau_n = \sum_{m=1}^{n} \frac{\alpha_m}{m} \tag{4.7e}$$

and $\gamma = 0.577\,215\,66\ldots$ is the Euler–Mascheroni constant. Note that, due to equation (4.3b), when $n \to \infty$ in equation (4.7e), the limit τ exists. Substituting equations (4.7a) and (4.7b) into equation (4.4a), expanding then in Taylor series the term $\frac{z}{z-1}$ for $|z| > 1$, and cutting the series beyond order nk we come to

$$g_k(z) \approx \frac{z^{nk}}{n^k} \zeta_n^k \sum_{m=0}^{nk} \frac{(k)_m}{m!} \frac{1}{z^m} \left(1 + \mathcal{O}\left(\frac{1}{z^{nk}}\right)\right) \qquad \text{for} \quad |z| > 1 \tag{4.8a}$$

with

$$(k)_m = k(k+1)\cdots(k+m-1) = \frac{\Gamma(k+m)}{\Gamma(k)}.$$

While substituting equations (4.7c)–(4.7e) into equation (4.4a) gives us

$$g_k(1) \approx (\ln n + \gamma - \tau_n)^k \eta_n^k. \tag{4.8b}$$

We replace equation (4.8) into equation (4.5b) to obtain

$$\Delta^r g_k(0) \approx (-1)^{r-1} r \left[g_k(1) - \lambda_n^{(k)} \right] + \frac{r!}{n^k} \zeta_n^k \sum_{l=0}^{nk} \frac{(k)_l}{l!} \mathcal{T}_{nk-l}^{(r)} \tag{4.9}$$

where we have used equation (A.4) to introduce the Stirling numbers of the second kind $\mathcal{T}_p^{(l)}$, and

$$\lambda_n^{(k)} \equiv \frac{\zeta_n^k}{n^k} \sum_{l=0}^{nk} \frac{(k)_l}{l!}.$$

The summation for $\lambda_n^{(k)}$ can be found in summation tables yielding [15]

$$\sum_{l=0}^{nk} \frac{(k)_l}{l!} = \frac{[k(n+1)]!}{k!(nk)!}.$$

Next, applying Stirling's approximation ($s! \approx \sqrt{2\pi s} s^s \mathrm{e}^{-s}$ for $s \gg 1$) to the factorials for $n \gg 1$, we find $\lambda \sim \mathrm{e}^k$, while $g_k(1) \approx \mathrm{e}^{k\{\ln[\ln n + \gamma - \tau_n] + \ln \eta_n\}}$; thus, λ is exponentially small as compared to $g_k(1)$, and can be neglected in equation (4.9). Replacing equation (4.9) back into equation (4.5*a*), using properties (A.5), and substituting in equation (4.4*b*), we find for the asymptotic distribution of the number of attractors

$$\rho_n(k) \approx \frac{\sqrt{2\pi n}}{k! 2^k} g_k(1) \sum_{l=n}^{nk} \frac{(-1)^{l-1}}{(l-1)!} \mathcal{S}_k^{(n)} + \mathcal{N}_n(k) \tag{4.10}$$

where

$$\mathcal{N}_n(k) = \frac{\sqrt{2\pi n}}{k! 2^k} \frac{1}{n^k} \zeta_n^k \frac{(k)_{[n(k-1)]}}{[n(k-1)]!}. \tag{4.11}$$

We show now that $\mathcal{N}_n(k)$ is a null term, that is to say

$$\lim_{n\to\infty} \sum_{k=1}^{n} k^m \mathcal{N}_n(k) = 0 \tag{4.12}$$

for any $m \geqslant 0$, and therefore it does not contribute to the averages taken with $\rho_n(k)$, for $n \gg 1$. First note that $\mathcal{N}_n(k) \sim \frac{1}{\sqrt{n}}$ for any k, so that in the summation (4.12) the terms with $k \sim \mathcal{O}(1)$ do not contribute. Hence, we can make the approximation for $n \gg 1$ and $k \gg 1$ in equation (4.11), and calculate equation (4.12) with it, obtaining

$$\mathcal{N}_n(k) \approx \frac{\mathrm{e}^{-(1+\frac{1}{2n})}}{\sqrt{n}} \frac{1}{k!} \left[\frac{1}{2}\zeta_n \mathrm{e}^{1+\frac{1}{2n}}\right]^k.$$

Since

$$\sum_{k=1}^{\infty} k^m \frac{x^k}{k!} = \mathcal{P}_m(x)\, \mathrm{e}^x$$

where $\mathcal{P}_m(x)$ is a polynomial in x of degree m, we have

$$\sum_{k=1}^{n} k^m \mathcal{N}_n(k) \approx \frac{\mathrm{e}^{-(1+\frac{1}{2n})}}{\sqrt{n}} \mathcal{P}_m\left(\frac{1}{2}\zeta_n\, \mathrm{e}^{1+\frac{1}{2n}}\right) \exp\left(\frac{1}{2}\zeta_n\, \mathrm{e}^{1+\frac{1}{2n}}\right)$$

and thus the expression goes to zero as $\frac{1}{\sqrt{n}}$, which proves equation (4.12).

Then, (4.10) can be expressed very simply in terms of Stirling numbers of the first kind by noting the following. Let us take for a moment $\alpha_m = 0$ in equation (4.4*a*), and also set $p = \infty$ with $|z| < 1$ (so the infinite sum converges), obtaining

$$\tilde{g}_k(z) = \left(\sum_{m=1}^{\infty} \frac{1}{m} z^m\right)^k = \{-\ln(1-z)\}^k$$

where the tilde indicates that we have set $\alpha_m = 0$. Using equation (A.2) and substituting in equation (4.4*b*) we arrive to

$$\tilde{\rho}_n(k) = \frac{\mathrm{e}^n}{n^n 2^k} (-1)^{n-k} \mathcal{S}_n^{(k)}.$$

Comparing with equation (4.10), we obtain, up to the null term, that

$$\rho_n(k) \approx \frac{e^n}{n^n}(-1)^{n-k}\mathcal{S}_n^{(k)}\left(\frac{\mu_n}{2}\right)^k \tag{4.13a}$$

where

$$\mu_n = 1 - \frac{\tau_n}{\gamma + \ln n}. \tag{4.13b}$$

Since $\rho_n(k)$ is a statistical distribution,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\rho_n(k) = 1.$$

This allows us to compute $\tau = \lim_{n\to\infty}\tau_n$ in equation (4.13*b*), by using property (A.1) of Stirling numbers, giving

$$\tau = \ln 2. \tag{4.13c}$$

We can replace equation (4.13*c*) back into equations (4.13*a*) and (4.13*b*), yielding an error of order $\mathcal{O}(1/\ln n)$, for $1 \leqslant k \leqslant n$. So we have, for the asymptotic approximation of the distribution of the number of attractors, the following expression

$$\rho_n(k) \approx \frac{e^n}{n^n}(-1)^{n-k}\mathcal{S}_n^{(k)}\left(\frac{\mu}{2}\right)^k\left(1 + \mathcal{O}\left(\frac{1}{\ln n}\right)\right) \tag{4.14a}$$

where

$$\mu = 1 - \frac{\ln 2}{\ln n}. \tag{4.14b}$$

Now we can use equation (4.14) to calculate averages. Using expression (A.3), we have

$$\Theta(n) \approx \frac{1}{2}\left(\ln 2n + \gamma + \mathcal{O}\left(\frac{1}{\ln n}\right)\right) \tag{4.15}$$

for the average number of attractors $\Theta(n)$, which coincides, as expected, with Kruskal asymptotic approximation (1.2). The variance can also be computed by using the derivative of (A.3)

$$\sigma^2 \approx \Theta(n)\left(1 + \mathcal{O}\left(\frac{1}{\ln n}\right)\right) \tag{4.16}$$

which is a new result.

5. Conclusion

We have proposed a new and simpler expression for the distribution of the number of attractors in the random map model (equation (3.3)). The number of operations involved for the numerical evaluation of the distribution grows exponentially with n, thus making expression (3.3) useless for direct calculations for $n \gg 1$. To overcome this difficulty, however, we have derived an asymptotic formula (equation (4.14)), from which we directly deduced asymptotic values both for the average number of attractors and for its variance (equations (4.15) and (4.16), respectively).

In the random map model, additional statistical figures are of interest, among which are the average attractor size, the average period (or average length) of the cycles; also, given a point x in the phase space, the expected length of its orbit, the expected length of the cycle in the attractor containing x, the expected number of points from which x can be attained, etc. Some of these figures have already been computed (for example, see [2, 3, 10]), and others are the subject of recent studies [16].

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Appendix. Properties of Stirling numbers

Here we list the equations and definitions on Stirling numbers necessary to follow the calculations in this paper. For a more extensive treatment see [11–13]. Stirling numbers of the first kind are generated by the functions

$$z(z+1)\cdots(z+n-1) = \sum_{m=1}^{n}(-1)^{n-m}\mathcal{S}_n^{(m)}z^m \tag{A.1}$$

and

$$\{\ln(1+z)\}^k = k!\sum_{r=k}^{\infty}\frac{\mathcal{S}_r^{(k)}}{r!}z^r \qquad \text{for} \quad |z|<1. \tag{A.2}$$

Applying the operator $z\frac{\mathrm{d}}{\mathrm{d}z}$ to equation (A.1) we obtain the important relation

$$z(z+1)\cdots(z+n-1)\sum_{m=1}^{n}\frac{z}{z+m-1} = \sum_{m=1}^{n}(-1)^{n-m}\mathcal{S}_n^{(m)}mz^m. \tag{A.3}$$

Stirling numbers of the first kind may be expressed in closed form by

$$\mathcal{S}_k^{(n)} = \sum_{l=0}^{k-n}(-1)^l\binom{k-1+l}{k-n+l}\binom{2k-n}{k-n-l}\mathcal{T}_{k-n+l}^{(1)}$$

where

$$\mathcal{T}_p^{(1)} = \frac{1}{l!}\sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}k^p \tag{A.4}$$

are the Stirling numbers of the second kind. It is known that Stirling numbers verify the relations

$$\mathcal{S}_k^{(n)} = 0 \qquad \text{if} \quad k<n \tag{A.5a}$$

$$\mathcal{T}_p^{(1)} = 0 \qquad \text{if} \quad p<l \tag{A.5b}$$

and

$$\sum_{k=m}^{n}\mathcal{S}_k^{(m)}\mathcal{T}_n^{(k)} = \sum_{k=m}^{n}\mathcal{S}_n^{(k)}\mathcal{T}_k^{(m)} = \delta_{m,n}. \tag{A.5c}$$

By means of Stirling numbers of the first kind it is possible to express derivatives of a function in terms of finite differences by the formula

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m}f(z) = m!\sum_{k=m}^{\infty}\frac{\mathcal{S}_k^{(m)}}{k!}\Delta^k f(z) \tag{A.6a}$$

if the summation is convergent, and where $\Delta f(x) = f(x+1) - f(x)$ and

$$\Delta^k f(z) = \sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}f(z+l). \tag{A.6b}$$

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